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ON WEAKLY 2-ABSORBING IDEALS OF COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring with identity $1 \neq 0$. Various generalizations of prime ideals have been studied. For example, a proper ideal I of R is*weakly prime* if $a, b \in R$ with $0 \neq ab \in I$, then either $a \in I$ or $b \in I$. Also a proper ideal I of R is said to be *2-absorbing* if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. In this paper, we introduce the concept of a *weakly 2-absorbing ideal.* A proper ideal I of R is called *a weakly* 2*-absorbing ideal of* R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. For example, every proper ideal of a quasi-local ring (R, M) with $M^3 = \{0\}$ is a weakly 2-absorbing ideal of R. We show that a weakly 2-absorbing ideal I of R with $I^3 \neq 0$ is a 2-absorbing ideal of R. We show that every proper ideal of a commutative ring R is a weakly 2-absorbing ideal if and only if either R is a quasi-local ring with maximal ideal M such that $M^3 = \{0\}$ or R is ring-isomorphic to $R_1 \times F$ where R_1 is a quasi-local ring with maximal ideal M such that $M^2 = \{0\}$ and F is a field or R is ring-isomorphic to $F_1 \times F_2 \times F_3$ for some fields F_1, F_2, F_3.

1. Introduction

In this paper, we study weakly 2-absorbing ideals in commutative rings with identity, which are a generalization of weakly prime ideals. Recall that 2-absorbing ideals, which are a generalization of prime ideals, were introduced and investigated in [4] and most recently in [3]. Recall from [2] that a proper ideal I of a commutative ring R is said to be a *weakly prime ideal of* R if whenever $a, b \in R$ and $0 \neq ab \in I$, then either $a \in I$ or $b \in I$. Also recall from [4] that a proper ideal I

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of a commutative ring R is called a 2-*absorbing ideal of* R if whenever $a, b, c \in R$ and $abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. We define a proper ideal of a commutative ring R to be a *weakly 2-absorbing ideal of* R if whenever $a, b, c \in R$ and $0 \neq abc \in I$, then either $ab \in I$ or $ac \in I$ or $bc \in I$. In the second section of this paper, many basic properties of weakly 2-absorbing ideals are studied, and in the third section, we characterize all commutative rings with the property that all proper ideals are weakly 2-absorbing ideals.

We assume throughout that all rings are commutative with $1 \neq 0$. Let R be a ring. Then $Nil(R)$ denotes the ideal of nilpotent elements of R. An ideal I of R is said to be a *proper ideal of* R if $I \neq R$. As usual, $\mathbb{Z}$, and $\mathbb{Z}_n$ will denote integers, and integers modulo n, respectively. Some of our examples use the $R(+)M$ construction as in [5]. Let R be a ring and M an R-module. Then $R(+)M = R \times M$ is a ring with identity $(1, 0)$ under addition defined by $(r, m) + (s, n) = (r + s, m + n)$ and multiplication defined by $(r, m)(s, n) = (rs, rn + sm)$. Note that $(0(+)M)^2 = 0$; so $0(+)M \subseteq Nil(R(+)M)$.

2. Basic properties of weakly 2-absorbing ideals

It is clear that every 2-absorbing ideal of a ring R is a weakly 2-absorbing ideal of R. If R is any commutative ring, then $I = \{0\}$ is a weakly 2-absorbing ideal of R by definition. If $I = \{0\}$, then I is a 2-absorbing ideal of $\mathbb{Z}_4$, but I is a weakly 2-absorbing ideal of $\mathbb{Z}_8$ that is not a 2-absorbing ideal of $\mathbb{Z}_8$. The following is an example of a nonzero weakly 2-absorbing ideal that is not a 2-absorbing ideal (also see Theorem 2.9 and Theorem 2.13).

Example 2.1. *Let $M = \{0, 4\}$. Then M is an ideal of $\mathbb{Z}_8$. Let $R = \mathbb{Z}_8(+)M$ and let $I = \{(0, 0), (0, 4)\}$. Since $abc \in I$ for some $a, b, c \in R \setminus I$ if and only if $abc = (0, 0)$, we conclude that I is a weakly 2-absorbing ideal of R. Since $(2, 0)(2, 0)(2, 0) \in I$ and $(4, 0) \notin I$, I is not a 2-absorbing ideal of R. For an infinite weakly 2-absorbing ideal that is not a 2-absorbing ideal, let M be as above and $K = M[X]$. Then K is an infinite ideal of $\mathbb{Z}_8[X]$. Let $R = \mathbb{Z}_8(+)K$ and let $I = \{0\}(+)K$. Then I is an infinite ideal of R. Again, since $abc \in I$ for some $a, b, c \in R \setminus I$ if and only if $abc = (0, 0)$, I is a weakly 2-absorbing ideal of R.*

We start with the following trivial lemma that we omit its proof.

Lemma 2.2. *If P_1 and P_2 are two distinct weakly prime ideals of a commutative ring R, then $P_1 \cap P_2$ is a weakly 2-absorbing ideal of R.*

Let I be a weakly 2-absorbing ideal of a ring R and $a, b, c \in R$. We say (a, b, c) is *a triple-zero of* I if $abc = 0$, $ab \notin I$, $bc \notin I$, and $ac \notin I$.

Theorem 2.3. *Let I be a weakly 2-absorbing ideal of a ring R and suppose that (a, b, c) is a triple-zero of I for some $a, b, c \in R$. Then*

(1) $abI = bcI = acI = \{0\}$.
(2) $aI^2 = bI^2 = cI^2 = \{0\}$.

Proof. **(1)**. Suppose that $abi \neq 0$ for some $i \in I$. Then $ab(c + i) \neq 0$. Since $ab \notin I$, we conclude that either $a(c + i) \in I$ or $b(c + i) \in I$, and hence $ac \in I$ or $bc \in I$, a contradiction. Thus $abI = \{0\}$. Similarly, one can show that $bcI = acI = \{0\}$.

(2). Suppose that $ai_1i_2 \neq 0$ for some $i_1, i_2 \in I$. Since $abI = acI = bcI = \{0\}$ by (1), we conclude that $a(b+i_1)(c+i_2) = ai_1i_2 \neq 0$. Hence either $a(b+i_1) \in I$ or $a(c+i_2) \in I$ or $(b+i_1)(c+i_2) \in I$, and thus either $ab \in I$ or $ac \in I$ or $bc \in I$, a contradiction. Thus $aI^2 = \{0\}$. Similarly, $bI^2 = cI^2 = \{0\}$. □

Theorem 2.4. *Let I be a weakly 2-absorbing ideal of R that is not a 2-absorbing ideal. Then $I^3 = \{0\}$.*

Proof. Since I is not a 2-absorbing ideal of R, I has a triple-zero (a, b, c) for some $a, b, c \in R$. Suppose that $i_1i_2i_3 \neq 0$ for some $i_1, i_2, i_3 \in I$. Then by Theorem 2.3 we have $(a + i_1)(b + i_2)(c + i_3) = i_1i_2i_3 \neq 0$. Hence either $(a + i_1)(b + i_2) \in I$ or $(a + i_1)(c + i_3) \in I$ or $(b + i_2)(c + i_3) \in I$, and thus either $ab \in I$ or $ac \in I$ or $bc \in I$, a contradiction. Hence $I^3 = \{0\}$. □

Corollary 2.5. *Let I be a weakly 2absorbing ideal of R. If I is not a 2-absorbing ideal of R, then $I \subseteq Nil(R)$.*

It should be noted that a proper ideal I of R with $I^3 = 0$ need not be a weakly 2-absorbing ideal of R. We have the following example.

Example 2.6. $R = \mathbb{Z}_{16}$. *Then $I = \{0, 8\}$ is an ideal of $\mathbb{Z}_{16}$ and $I^3 = 0$, but* $2.2.2 = 8 \in I$ *and* $4 \notin I$.

Theorem 2.7. *Let I be a weakly 2-absorbing ideal of R that is not a 2-absorbing ideal. Then*

(1) *If $w \in Nil(R)$, then either $w^2 \in I$ or $w^2I = wI^2 = \{0\}$.*
(2) $Nil(R)^2I^2 = \{0\}$.

Proof. **(1)**. Let $w \in Nil(R)$. First, we show that if $w^2I \neq \{0\}$, then $w^2 \in I$. Hence assume that $w^2I \neq \{0\}$. Let n be the least positive integer such that $w^n = 0$. Then $n \geq 3$ and for some $i \in I$ we have $w^2(i + w^{n-2}) = w^2i \neq 0$. Hence either $w^2 \in I$ or $(wi + w^{n-1}) \in I$. If $w^2 \in I$, then we are done. Thus assume $(wi+w^{n-1}) \in I$. Hence $w^{n-1} \in I$ and $w^{n-1} \neq 0$, and thus $w^2 \in I$. Hence for each

$w \in Nil(R)$, we have either $w^2 \in I$ or $w^2 I = \{0\}$. Now assume that $v^2 \notin I$ for some $v \in Nil(R)$. Then $v^2 I = \{0\}$. We will show that $vI^2 = \{0\}$. Assume that $vi_1 i_2 \neq 0$ for some $i_1, i_2 \in I$. Let m be the least positive integer such that $v^m = 0$. Since $v^2 \notin I$, $m \geq 3$ and $v^2 I = 0$. Hence $v(v + i_1)(v^{m-2} + i_2) = vi_1 i_2 \neq 0$. Since $0 \neq v(v+i_1)(v^{m-2}+i_2) \in I$, one can conclude that either $v^2 \in I$ or $v^{m-1} \neq 0$ and $v^{m-1} \in I$. Hence in both cases, we have $v^2 \in I$, a contradiction. Thus $vI^2 = \{0\}$.

(2). Let $a, b \in Nil(R)$. If either $a^2 \notin I$ or $b^2 \notin I$, then $abI^2 = \{0\}$ by (1). Hence suppose that $a^2 \in I$ and $b^2 \in I$. Then $ab(a + b) \in I$. If $(a, b, a + b)$ is a triple-zero of I, then $abI = \{0\}$ by Theorem 2.3(1), and hence $abI^2 = \{0\}$. if $(a, b, a+b)$ is not a triple-zero of I, then one can easily see that $ab \in I$, and hence $abI^2 = \{0\}$ by Theorem 2.4. □

Corollary 2.8. *Suppose that A, B, C are weakly 2-absorbing ideals of a ring R such that none of them is a 2-absorbing ideal of R. Then $A^2BC = AB^2C = ABC^2 = A^2B^2 = A^2C^2 = B^2C^2 = \{0\}$*

If I is a 2-absorbing ideal of a ring R, then there are at most two prime ideals of R that are minimal over I (see [4, Theorem 2.3] and [3, Theorem 2.5]). In the following result, we show that for every $n \geq 2$, there is a ring R and a nonzero weakly 2-absorbing ideal I of R such that there are exactly n prime ideals of R that are minimal over I.

Theorem 2.9. *Let $n \geq 2$. Then there is a ring R and a nonzero weakly 2-absorbing I ideal of R such that there are exactly n prime ideals of R that are minimal over I.*

Proof. Let $n \geq 2$ and $D = \mathbb{Z}_8 \times \cdots \times \mathbb{Z}_8$ (n times). Let $M = \{0, 4\}$ an ideal of $\mathbb{Z}_8$. For every $x = (a_1, \ldots, a_n) \in D$, define $xM = a_1 M$. Then M is a $D-module$. Now consider the idealization ring $R = D(+)M$ and $I = \{(0, \ldots, 0)\}(+)M$. We note that if $a, b, c \in R \setminus I$ and $abc \in I$, then $abc = ((0, \ldots, 0), 0)$. Hence I is a nonzero weakly 2-absorbing ideal of R. Since every prime ideal of R is of the form $P(+)M$ for some prime ideal P of D by [5, Theorem 25.1 (3)], we conclude that there are exactly n prime ideals of R that are minimal over I. □

Theorem 2.10. *Let $R = R_1 \times R_2$ be a decomposable commutative ring and I be a proper ideal of R_1. The following statements are equivalent:*

(1) *$I \times R_2$ is a weakly 2-absorbing ideal of R.*
(2) *$I \times R_2$ is a 2-absorbing ideal of R.*
(3) *I is a 2-absorbing ideal of R_1.*

Proof. (1) $\Rightarrow$ (2). Since $I \times R_2 \not\subseteq Nil(R)$, $I \times R_2$ must be a 2-absorbing ideal of R by Corollary 2.5. (2) $\Rightarrow$ (3). The claim is clear. (3) $\Rightarrow$ (1). If I is a 2-absorbing ideal of R_1, then it is easily verified that $I \times R_2$ is a 2-absorbing ideal of R, and thus $I \times R_2$ is a weakly 2-absorbing ideal of R. $\square$

Theorem 2.11. *Let $R = R_1 \times R_2$ where R_1 and R_2 are commutative rings with identity. Let I_1 be a nonzero proper ideal of R_1 and J be a nonzero ideal of R_2. The following statements are equivalent:*

(1) *$I \times J$ is a weakly 2-absorbing ideal of R.*
(2) *$J = R_2$ and I is a 2-absorbing ideal of R_1 or J is a prime ideal of R_2 and I is a prime ideal of R_1.*
(3) *$I \times J$ is a 2-absorbing ideal of R.*

Proof. (1) $\Rightarrow$ (2). Suppose that $I \times J$ is a weakly 2-absorbing ideal of R. If $J = R_2$, then I is a 2-absorbing ideal of R_1 by Theorem 2.10. Suppose that $J \neq R_2$. We show that J is a prime ideal of R_2 and I is a prime ideal of R_1. Let $a, b \in R_2$ such that $ab \in J$, and let $0 \neq i \in I$. Then $(i,1)(1,a)(1,b) = (i,ab) \in I \times J \setminus \{(0,0)\}$. Since $(1,a)(1,b) = (1,ab) \notin I_1 \times J$, we conclude that either $(i,1)(1,a) = (i,a) \in I \times J$ or $(i,1)(1,b) = (i,b) \in I \times J$, and hence either $a \in J$ or $b \in J$. Thus J is a prime ideal of R_2. Similarly, let $c, d \in R_1$ such that $cd \in I$, and let $0 \neq j \in J$. Then $(c,1)(d,1)(1,j) = (cd,j) \in I \times J \setminus \{(0,0)\}$. Since $(c,1)(d,1) = (cd,1) \notin I \times J$, we conclude that either $(c,1)(1,j) = (c,j) \in I \times J$ or $(d,1)(1,j) = (d,j) \in I \times J$, and thus either $c \in I$ or $d \in I$. Hence I ia a prime ideal of R_1. (2) $\Rightarrow$ (3). If $J = R_2$ and I is a 2-absorbing ideal of R_1, then $I \times R_2$ is a 2-absorbing ideal of R by Theorem 2.10. Suppose that I is a prime ideal of R_1 and J is a prime ideal of R_2. Suppose that $(a_1,b_1)(a_2,b_2)(a_3,b_3) \in I \times J$ for some $a_1, a_2, a_3 \in R_1$ and for some $b_1, b_2, b_3 \in R_2$. Then at least one of the $a_i's$ is in I, say a_1, and at least one of the $b_i's$ is in J, say b_2. Thus $(a_1,b_1)(a_2,b_2) \in I \times J$. Hence $I \times J$ is a 2-absorbing ideal of R. (3) $\Rightarrow$ (1). No comments. $\square$

The following example shows that the hypothesis that J ia a nonzero ideal of R_2 in Theorem 2.11 is crucial.

Example 2.12. *Let $R_1 = \mathbb{Z}_8(+)M$ and $I = \{0\}(+)M$ as in example 2.1. Let R_2 be a field. Then $I \times \{0\}$ is a weakly 2-absorbing ideal of $R_1 \times R_2$ that is not a 2-absorbing ideal of $R_1 \times R_2$. Observe that I is not a prime ideal of R_1.*

Theorem 2.13. *Let $R = R_1 \times R_2$ be a commutative ring. Let I be a nonzero proper ideal of R_1 and J be an ideal of R_2. The following statements are equivalent:*

(1) *$I \times J$ is a weakly 2-absorbing ideal of R that is not a 2-absorbing ideal.*
(2) *I is a weakly prime ideal of R_1 that is not a prime ideal and $J = \{0\}$ is a prime ideal of R_2.*

Proof. (1) $\Rightarrow$ (2). Assume that $I \times J$ is a weakly 2-absorbing ideal of R that is not a 2-absorbing ideal. Suppose that $J \neq \{0\}$. Then $I \times J$ is a 2-absorbing ideal of R by Theorem 2.11, which contradicts the hypothesis. Thus $J = \{0\}$. We show that $J = \{0\}$ is a prime ideal of R_2 (and hence R_2 is an integral domain). Suppose that $ab \in J = \{0\}$ for some $a, b \in R_2$. Let $0 \neq i \in I$. Since $(i,1)(1,a)(1,b) = (i,ab) \in I \times J \setminus \{(0,0)\}$ and $(1,a)(1,b) = (1,ab) \notin I \times J$, we conclude that either $(i,1)(1,a) = (i,a) \in I \times J$ or $(i,1)(1,b) = (i,b) \in I \times J$, and thus $a \in J$ or $b \in J$. Hence $J = \{0\}$ is a prime ideal of R_2. We show that I is a weakly prime ideal of R_1. Suppose that $ab \in I \setminus \{0\}$ for some $a, b \in R_1$. Since $(a,1)(b,1)(1,0) = (ab,0) \in I \times \{0\} \setminus \{(0,0)\}$ and $(a,1)(b,1) = (ab,1) \notin I \times \{0\}$, we conclude that either $(a,1)(1,0) = (a,0) \in I \times \{0\}$ or $(b,1)(1,0) = (b,0) \in I \times \{0\}$, and thus either $a \in I$ or $b \in I$. Hence I is a weakly prime ideal of R_1. If I is a prime ideal of R_1, then it is easily verified that $I \times \{0\}$ is a 2-absorbing ideal of R, which is a contradiction. (2) $\Rightarrow$ (1). Suppose that I is a weakly prime ideal of R_1 that is not a prime ideal and $J = \{0\}$ is a prime ideal of R_2. We show that $I \times \{0\}$ is a weakly 2-absorbing ideal of R. Suppose that $(a,b)(c,d)(e,f) = (ace, bdf) \in I \times \{0\} \setminus \{(0,0)\}$. Since I is a weakly prime of R_1, we may assume $a \in I$. Since R_2 is an integral domain, we may assume $d = 0$. Hence $(a,b)(c,d) = (a,b)(c,0) = (ac,0) \in I \times \{0\}$. Thus $I \times \{0\}$ is a weakly 2-absorbing ideal of R. We show that $I \times \{0\}$ is not a 2-absorbing ideal of R. Since I is a weakly prime ideal of R_1 that is not a prime ideal, there are $a, b \in R_1$ such that $ab = 0$ but neither $a \in I$ nor $b \in I$. Since $(a,1)(b,1)(1,0) = (0,0)$ and neither $(a,1)(b,1) = (ab,1) \in I \times \{0\}$ nor $(a,1)(1,0) = (a,0) \in I \times \{0\}$ nor $(b,1)(1,0) = (b,0) \in I \times \{0\}$, we conclude that $I \times \{0\}$ is not a 2-absorbing ideal of R. $\square$

Let R_1, R_2 and R_3 be commutative rings with identity and set $R = R_1 \times R_2 \times R_3$. An ideal I of R will have the form $I_1 \times I_2 \times I_3$ where I_1, I_2 and I_3 are ideals of R_1, R_2 and R_3, respectively. The next two theorems show that weakly 2-absorbing ideals are really of interest in rings of this form.

Theorem 2.14. *Let $R = R_1 \times R_2 \times R_3$ where R_1, R_2 and R_3 are commutative rings with identity. If I is a weakly 2-absorbing ideal of R, then either $I = \{(0,0,0)\}$, or I is a 2-absorbing ideal of R.*

Proof. Since $\{0\}$ is a weakly 2-absorbing ideal in any ring, we may assume that $I = I_1 \times I_2 \times I_3 \neq \{(0,0,0)\}$. Since $I \neq \{(0,0,0)\}$, there is an element $(0,0,0) \neq (a,b,c) \in I$. Then $(a,1,1)(1,b,1)(1,1,c) = (a,b,c)$, and hence either $(a,b,1) \in I$ or $(a,1,c) \in I$ or $(1,b,c) \in I$. If $(a,b,1) \in I$, then $I_3 = R_3$. Likewise if $(a,1,c) \in I$ or $(1,b,c) \in I$, then $I_2 = R_2$ or $I_1 = R_1$, respectively. So $I = I_1 \times I_2 \times R_3$ or $I = I_1 \times R_2 \times I_3$ or $I = R_1 \times I_2 \times I_3$. Hence $I \not\subseteq Nil(R)$. Since I is a weakly 2-absorbing ideal of R and $I \not\subseteq Nil(R)$, I is a 2-absorbing ideal of R by Corollary 2.5. □

Theorem 2.15. *Let $R = R_1 \times R_2 \times R_3$ where R_1, R_2 and R_3 are commutative rings with identity. Let I_1 be a proper ideal of R_1, I_2 be an ideal of R_2, and I_3 be an ideal of R_3 such that $L = I_1 \times I_2 \times I_3 \neq \{(0,0,0)\}$. The following statements are equivalent:*

(1) *$L = I_1 \times I_2 \times I_3$ is a weakly 2-absorbing ideal of R.*
(2) *$L = I_1 \times I_2 \times I_3$ is a 2-absorbing ideal of R.*
(3) *$L = I_1 \times R_2 \times R_3$ and I_1 is a 2-absorbing ideal of R_1 or $L = I_1 \times I_2 \times R_3$ such that I_1 is a prime ideal of R_1 and I_2 is a prime ideal of R_2 or $L = I_1 \times R_2 \times I_3$ such that I_1 is a prime ideal of R_1 and I_3 is a prime ideal of R_3.*

Proof. (1) $\Rightarrow$ (2). Since L is a nonzero weakly 2-absorbing ideal, L is a 2-absorbing ideal of R by Theorem 2.14. (2) $\Rightarrow$ (3). Since L is a 2-absorbing ideal of R, I_1 is a 2-absorbing ideal of R_1. Since I_1 is a proper ideal of R_1, by the proof of Theorem 2.14 either $I_2 = R_2$ or $I_3 = R_3$. Assume that $I_2 \neq R_2$ and $I_3 = R_3$. We show that I_1 is a prime ideal of R_1 and I_2 is a prime of R_2. Let $a, b \in R_1$ such that $ab \in I_1$, and let $c, d \in R_2$ such that $cd \in I_2$. Then $(a,1,1)(1,cd,1)(b,1,1) = (ab,cd,1) \in L \setminus \{(0,0,0)\}$. Since $(a,1,1)(b,1,1) \notin L$, we have $(a,1,1)(1,cd,1) = (a,cd,1) \in L$ or $(1,cd,1)(b,1,1) = (b,cd,1) \in L$, and hence $a \in I_1$ or $b \in I_1$. Thus I_1 is a prime ideal of R_1. Similarly, since $(ab,1,1)(1,c,1)(1,d,1) = (ab,cd,1) \in L \setminus \{(0,0,0)\}$ and $(1,c,1)(1,d,1) = (1,cd,1) \notin L$, we conclude that either $(ab,1,1)(1,c,1) = (ab,c,1) \in L$ or $(ab,1,1)(1,d,1) = (ab,d,1) \in L$, and hence either $c \in I_2$ or $d \in I_2$. Thus I_2 is a prime ideal of R_2. Finally, assume $I_2 = R_2$ and $I_3 \neq R_3$. By an argument similar to that we applied on the ideal $I_1 \times I_2 \times R_3$, we conclude that I_1 is a prime ideal of R_1 and I_3 is a prime ideal of R_3. (3) $\Rightarrow$ (1). If L is one of the given three forms, then it is easily verified that L is a 2-absorbing ideal of R, and hence L is a weakly 2-absorbing ideal of R. □

Theorem 2.16. *Let A be a weakly 2-absorbing ideal of a commutative ring R. Then:*

(1) *If I is an ideal of R with $I \subseteq A$, then A/I is a weakly 2-absorbing ideal of R/I.*
(2) *If R_0 is a subring of R, then $A \cap R_0$ is a weakly 2-absorbing ideal of R_0.*
(3) *If S is a multiplicatively closed subset of R with $A \cap S = \emptyset$, then A_S is a weakly 2-absorbing ideal of R_S.*

Proof. **(1)**. Let $\bar{R} = R/I$, $\bar{A} = A/I$, and pick $\bar{a}, \bar{b}, \bar{c} \in \bar{R}$ such that $0 \neq \bar{a}\bar{b}\bar{c} \in \bar{A}$. Since $\bar{a}\bar{b}\bar{c} \neq 0$, we have $abc \in R - I$. Hence $0 \neq abc \in A$. Since A is weakly 2-absorbing, we have $ab \in A$ or $ac \in A$ or $bc \in A$. Consequently, $\bar{a}\bar{b} \in \bar{A}$ or $\bar{a}\bar{c} \in \bar{A}$ or $\bar{b}\bar{c} \in \bar{A}$. **(2)**. The proof is straightforward. **(3)**. Suppose that $0 \neq (x/r)(y/s)(z/t) \in A_S$ where $x, y, z \in R$ and $r, s, t \in S$ but $(x/r)(y/s) \notin A_S$ and $(x/r)(z/s) \notin A_S$. Then $(xyz)/(rst) = a/u$ for some $a \in A$ and $u \in S$. So there exists $v \in S$ with $vuxyz = vrsta \in A$. Thus we have $0 \neq (vux)yz \in A$ but $(vux)y \notin A$ and $(vux)z \notin A$. Since A is weakly 2-absorbing, it follows that $yz \in A$, that is $(y/s)(z/t) \in A_S$. □

3. Rings with the property that all proper ideals are weakly 2-absorbing

For a commutative ring R, let $J(R)$ denotes the intersection of all maximal ideals of R.

Lemma 3.1. *Let R be a commutative ring and $a, b, c \in J(R)$. Then the ideal $abcR$ is a weakly 2-absorbing ideal of R if and only if $abc = 0$.*

Proof. Let $a, b, c \in J(R)$. If $abc = 0$, then $abcR$ is a weakly 2-absorbing ideal of R. Now suppose that $abc \neq 0$ and $abcR$ is a weakly 2-absorbing ideal of R. Since $abcR$ is a weakly 2-absorbing ideal of R and $0 \neq abc \in abcR$, we conclude that either $ab \in abcR$ or $ac \in abcR$ or $bc \in abcR$. Without lost of generality, we may assume that $ab \in abcR$. Thus $ab = abck$ for some $k \in R$, and hence $ab(1-ck) = 0$. Since $ck \in J(R)$, $1 - ck$ is a unit of R. Thus $ab(1 - ck) = 0$ implies that $ab = 0$, and thus $abc = 0$ which is a contradiction. Hence $abc = 0$. □

Theorem 3.2. *Let (R, M) be a quasi-local ring. Then every proper ideal of R is weakly 2-absorbing if and only if $M^3 = \{0\}$.*

Proof. Assume that every proper ideal of R is weakly 2-absorbing. Let $a, b, c \in M$. Since $abcR$ is a weakly 2-absorbing ideal of R, $abc = 0$ by Lemma 3.1. Thus $M^3 = \{0\}$. Conversely, assume that $M^3 = \{0\}$, and let I be a proper ideal of R such that $I \neq \{0\}$. Suppose that $abc \in I$ and $abc \neq 0$. Since $M^3 = \{0\}$ and $abc \neq 0$, a is a unit of R or b is a unit of R or c is a unit of R, and thus either $ab \in I$ or $ac \in I$ or $bc \in I$. Hence I is a weakly 2-absorbing ideal of R. □

Corollary 3.3. *Let (R, M) be a quasi-local ideal of R such that $M^2 = \{0\}$. Then every proper ideal of R is a 2-absorbing ideal of R.*

PROOF. Let I be a proper ideal of R and suppose that $abc \in I$ for some $a, b, c \in R$. Since $M^3 = \{0\}$, I is a weakly 2-absorbing ideal of R by Theorem 3.2. Hence if $abc \in I \setminus \{0\}$, then there is nothing to prove. Thus assume that $abc = 0$. Since $M^2 = \{0\}$ and $abc = 0$, either $ab = 0 \in I$ or $ac \in I$ or $bc = 0 \in I$. Thus I is a 2-absorbing ideal of R. $\square$

Theorem 3.4. *Let (R_1, M_1) and (R_2, M_2) be quasi-local commutative rings with maximal ideals M_1 and M_2 respectively, and let $R = R_1 \times R_2$. Then every proper ideal of R is a weakly 2-absorbing ideal of R if and only if $M_1^2 = M_2^2 = \{0\}$ and either R_1 or R_2 is a field.*

PROOF. Suppose that every proper ideal of R is a weakly 2-absorbing ideal of R. Let $a, b \in M_1$ and suppose that $ab \neq 0$. Then $I = abR_1 \times \{0\}$ is a weakly 2-absorbing ideal of R. Since $(a, 1)(b, 1)(1, 0) = (ab, 0) \in I \setminus \{(0, 0)\}$ and $(a, 1)(b, 1) \notin I$, either $(a, 1)(1, 0) = (a, 0) \in I$ or $(b, 1)(1, 0) = (b, 0) \in I$. Assume that $(a, 0) \in I$. Then $a = abk$ for some $k \in R_1$. Hence $a(1 - bk) = 0$. Since $1 - bk$ is a unit of R_1, $a = 0$ which is a contradiction. Also, if $(b, 0) \in I$, then one can conclude that $b = 0$ which is a contradiction again. Thus $M_1^2 = \{0\}$. Now assume $a, b \in M_2$ such that $ab \neq 0$. Then $I = \{0\} \times abR_2$ is a weakly 2-absorbing ideal of R. Since $(1, a)(1, b)(0, 1) = (0, ab) \in I$, by an argument similar to that we applied on M_1 we conclude that either $a = 0$ or $b = 0$ which is a contradiction. Thus $M_2^2 = \{0\}$. Suppose that R_1 is not a field. We show that R_2 is a field. Since R_1 is not a field, $M_1 \neq \{0\}$ and $J = M_1 \times \{0\}$ is a weakly 2-absorbing ideal of R. Suppose that R_2 is not a field. Since $M_2^2 = \{0\}$ and R_2 is not a field, there is a $c \in M_2$ such that $c \neq 0$ and $c^2 = 0$. Let $m \in M_1$ such that $m \neq 0$. Then $(m, 1)(1, c)(1, c) = (m, c^2) = (m, 0) \in J = M_1 \times \{0\} \setminus \{(0, 0)\}$, but neither $(m, 1)(1, c) = (m, c) \in J$ nor $(1, c)(1, c) = (1, c) \in J$, which is a contradiction. Hence $M_2 = \{0\}$, and thus R_2 is a field. Conversely, suppose that $M_1^2 = \{0\}$ and R_2 is a field. Since $M_1^2 = \{0\}$, every proper ideal of R_1 is a 2-absorbing ideal of R_1 by Corollary 3.3. Since $M_1^2 = \{0\}$ and R_2 is a field, the ideal $\{0\} \times R_2$ is a weakly 2-absorbing ideal of R. Since R_2 is a field, the ideal $R_1 \times \{0\}$ is a weakly 2-absorbing ideal of R. Let J be a proper ideal of R_1 such that $J \neq \{0\}$. Since J is a 2-absorbing ideal of R_1, $J \times R_2$ is a weakly 2-absorbing ideal of R by Theorem 2.10. Finally, we show that $I = J \times \{0\}$ is a weakly 2-absorbing ideal of R. Suppose that $(a_1, b_1)(a_2, b_2)(a_3, b_3) \in R \setminus \{(0, 0)\}$ for some $a_1, a_2, a_3 \in R_1$ and for some $b1, b_2, b_3 \in R_2$. Since $M_1^2 = \{0\}$, only one of the a_i's is in M_1, say $a_1 \in M_1$ and a_2, a_3 are units of R_1. Since $a_1 a_2 a_3 \in J$ and a_2, a_3 are units of R_1,

$a_1 \in J$. Since R_2 is a field and $b_1b_2b_3 = 0$, at least one of the b_i's is equal to 0, say $b_2 = 0$. Hence $(a_1, b_1)(a_2, 0) = (a_1a_2, 0) \in I$. Thus I is a weakly 2-absorbing ideal of R. □

Theorem 3.5. *Let R_1, R_2, and R_3 be commutative rings, and let $R = R_1 \times R_2 \times R_3$. Then every proper ideal of R is a weakly 2-absorbing ideal of R if and only if R_1, R_2, R_3 are fields.*

Proof. If R_1, R_2, and R_3 are fields, then by [4, Theorem 3.4(3)] every nonzero proper ideal of R is a 2-absorbing ideal of R, and hence every proper ideal of R is a weakly 2-absorbing ideal of R. Conversely, suppose that every proper ideal of R is a weakly 2-absorbing ideal of R and one of the $R_i's$, $1 \le i \le 3$, is not a field. Without lost of generality, we may assume R_1 is not a field. Hence R_1 has a proper ideal J such that $J \neq \{0\}$. Let $I = J \times \{0\} \times \{0\}$. Then I is a weakly 2-absorbing ideal of R. Let $m \in J$ such that $m \neq 0$. Then $(m, 1, 1)(1, 0, 1)(1, 1, 0) = (m, 0, 0) \in I \setminus \{(0, 0, 0)\}$ but neither $(m, 1, 1)(1, 0, 1) = (m, 0, 1) \in I$ nor $(m, 1, 1)(1, 1, 0) = (m, 1, 0) \in I$ nor $(1, 0, 1)(1, 1, 0) = (1, 0, 0) \in I$, which is a contradiction. Thus R_1, R_2, and R_3 are fields. □

Lemma 3.6. *Suppose that every proper ideal of R is a weakly 2-absorbing ideal. Then R has at most three maximal ideals.*

Proof. Suppose that M_1, M_2, M_3, M_4 are distinct maximal ideals of R. Let $I = M_1 \cap M_2 \cap M_3$. Since there are three prime ideals of R that are minimal over I, I is not a 2-absorbing ideal of R by [3, Theorem 2.5]. Hence I is a weakly 2-absorbing ideal of R that is not a 2-absorbing ideal of R. Thus $I^3 = \{0\}$ by Theorem 2.4. Hence $I^3 = M_1^3M_2^3M_3^3 = \{0\} \subset M_4$, and thus one of the M_i's, $1 \le i \le 3$, is contained in M_4, which is a contradiction. Hence R has at most three distinct maximal ideals. □

Theorem 3.7. *A commutative ring R has the property that every proper ideal is a weakly 2-absorbing ideal of R if and only if one of the following statements hold:*

(1) *(R, M) is a quasi-local ring with $M^3 = 0$.*
(2) *R is ring-isomorphic to $R_1 \times F$, where R_1 is a quasi-local ring with maximal ideal M such that $M^2 = \{0\}$ and F is a field.*
(3) *R is ring-isomorphic to $F_1 \times F_2 \times F_3$, where F_1, F_2, F_3 are fields.*

Proof. If R satisfies condition (1), then every proper ideal of R is a weakly 2-absorbing ideal of R by Theorem 3.2. If R satisfies condition (2), then every proper ideal of R is a weakly 2-absorbing ideal of R by Theorem 3.4. If R satisfies condition (3), then every proper ideal of R is a weakly 2-absorbing ideal of R by Theorem 3.5. Conversely, suppose that every proper ideal of R is a weakly 2-absorbing ideal. Then R has at most three maximal ideals by Lemma 3.6. Hence we consider the following three cases: **Case 1**. Suppose that R has exactly one maximal ideal, call it M. Then $M^3 = \{0\}$ by Theorem 3.2. **Case 2**. Suppose that R has exactly two maximal ideals, say M_1 and M_2 are the maximal ideals of R. Then $J(R) = M_1 \cap M_2$ is a weakly 2-absorbing ideal of R (in fact, $J(R)$ is a 2-absorbing ideal of R). We show $J(R)^3 = \{0\}$. Let $a, b, c \in J(R)$. Since $abcR$ is a weakly 2-absorbing ideal of R, we conclude that $abc = 0$ by Lemma 3.1. Thus $J(R)^3 = M_1^3 \cap M_2^3 = \{0\}$. Hence R is ring-isomorphic to $R/M_1^3 \times R/M_2^3$. Since R/M_1^3 and R/M_2^3 are quasi-local commutative rings, we conclude that R is ring-isomorphic to $R_1 \times F$, where R_1 is quasi-local ring with maximal ideal M such that $M^2 = \{0\}$ and F is a field by Theorem 3.4. **Case 3**. Suppose that R has exactly three maximal ideals, say M_1, M_2, M_3 are the maximal ideals of R. Hence $J(R) = M_1 \cap M_2 \cap M_3$ is a weakly 2-absorbing ideal of R. Since $J(R)$ is the intersection of three prime ideals of R, $J(R)$ is not a 2-absorbing ideal of R by [4]. Hence $J(R)^3 = \{0\}$ by Theorem 2.4. Since $J(R)^3 = M_1^3 \cap M_2^3 \cap M_3^3 = \{0\}$, we conclude that R is ring-isomorphic to $R/M_1^3 \times R/M_2^3 \times R/M_3^3$. Thus R is ring-isomorphic to $F_1 \times F_2 \times F_3$, where F_1, F_2, F_3 are fields by Theorem 3.5. $\square$

Corollary 3.8. *Let n be a positive integer. Then every proper ideal of $R = \mathbb{Z}_n$ is a weakly 2-absorbing ideal of R if and only if either $n = q^3$ for some prime positive integer q or $n = q^2p$ for some distinct prime positive integers q, p or $n = q_1q_2q_3$ for some distinct prime positive integers q_1, q_2, q_3.*

Let I be a 2-absorbing ideal of a commutative ring R and suppose that $I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, and I_3 of R. Then by [4] either $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$. We are unable to answer the following question:

Question. Suppose that I is a weakly 2-absorbing ideal of a commutative ring R that is not a 2-absorbing ideal and $0 \neq I_1I_2I_3 \subseteq I$ for some ideals I_1, I_2, and I_3 of R. Does it imply that $I_1I_2 \subseteq I$ or $I_1I_3 \subseteq I$ or $I_2I_3 \subseteq I$?

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